ULTRA-DISCRETIZATION OF THE $G_2^{(1)}$-GEOMETRIC CRYSTALS TO THE $D_4^{(3)}$-PERFECT CRYSTALS

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Abstract. We obtain the affirmative answer to the conjecture in [15]. More, precisely, let $\chi := (\mathcal{V}, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ be the affine geometric crystal of type $G_2^{(1)}$ in [14] and $\mathcal{UD}(\chi, T, \theta)$ a ultra-discretization of χ with respect to a certain positive structure θ. Then we show that $\mathcal{UD}(\chi, T, \theta)$ is isomorphic to the limit of coherent family of perfect crystals of type $D_4^{(3)}$ in [7].

1. Introduction

In [5], we introduced the notion of perfect crystal, which holds several nice properties, *e.g.,*, the existence of the isomorphism of crystals:

$$B(\lambda) \cong B(\sigma(\lambda)) \otimes B,$$

where B is a perfect crystal of level $l \in \mathbb{Z}_{>0}$, $B(\lambda)$ is the crystal of the integrable highest weight module of a quantum affine group with the level l highest weight λ and σ is a certain bijection on dominant weights. Iterating this isomorphism, one can get the so-called Kyoto path model for $B(\lambda)$, which plays an crucial role in calculating the one-point functions for vertex-type lattice models ([5],[6]).

In [6] perfect crystals with arbitrary level has been constructed explicitly for affine Kac-Moody algebra of type $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $D_{n+1}^{(2)}$, $A_{2n-1}^{(2)}$ and $A_{2n}^{(2)}$. In [16], the $G_2^{(1)}$ case has been accomplished. But, so far the other cases except $D_4^{(3)}$ have not yet been obtained. In the recent work [7], they constructed the perfect crystal of type $D_4^{(3)}$ with arbitrary level explicitly. A coherent family of perfect crystals is defined in [4] and it has been shown that the perfect crystals in [6] constitute a coherent family. A coherent family $\{B_l\}_{l\geq 1}$ of perfect crystals B_l possesses a limit B_∞ which still keeps a structure of crystal. This has a similar property to B_l, that is, there exists the isomorphism of crystals:

$$B(\infty) \cong B(\infty) \otimes B_\infty,$$

where $B(\infty)$ is the crystal of the nilpotent subalgebra $U_q^-(\mathfrak{g})$ of a quantum affine algebra $U_q(\mathfrak{g})$. An iteration of the isomorphism also produces a path model of $B(\infty)$([4]). It is shown in [7] that the obtained perfect crystals consists of a coherent family and the structure of the limit B_∞ has been described explicitly.

Geometric crystal is an object defined over certain algebraic (or ind-)variety which seems to be a kind of geometric lifting of Kashiwara's crystal. It is defined in [1] for reductive algebraic groups and is extended to general Kac-Moody

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cases in [13]. For a fixed Cartan data $(A, \{\alpha_i\}_{i\in I}, \{\alpha_i^\vee\}_{1\in I})$, a geometric crystal consists of an ind-variety X over the complex number $\mathbb{C}$, a rational $\mathbb{C}^\times$-action $e_i : \mathbb{C}^\times \times X \longrightarrow X$ and rational functions $\gamma_i, \varepsilon_i : X \longrightarrow \mathbb{C}$ $(i \in I)$, which satisfy the conditions as in Definition 2.1. It has many similarity to the theory of crystals, *e.g.,* some product structure, Weyl group actions, R-matrices, *etc.* Moreover, one has a direct connection between geometric crystals and free crystals, called tropicalization/ ultra-discretization procedure (see §2). Here let us explain this procedure. For an algebraic torus T' and a birational morphism $\theta : T' \to X$, the pair (T', θ) is positive if it satisfies the conditions as in Sect.2, roughly speaking: Through the morphism θ, we can induce a geometric crystal structure on T' from X and express the data e_i^c, γ_i and ε_i $(i \in I)$ using the coordinate of T' explicitly. In case each of them are expressed as a ratio of positive polynomials, it is said that (T', θ) is a positive structure of the geometric crystal $(X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$. Then by using a map $v : \mathbb{C}(c) \setminus \{0\} \to \mathbb{Z}$ $(v(f) := \deg(f))$, we can define a morphism $T' \to \mathbb{Z}^m$ $(m = \dim T' = \dim X)$, which defines the so-called ultra-discretization functor. If $\theta : T' \to X$ is a positive structure on X, then we obtain a Kashiwara's crystal from X by applying the ultra-discretization functor([1]).

Let G (resp. $\mathfrak{g} = \langle \mathfrak{t}, e_i, f_i\rangle_{i\in I}$) be the affine Kac-Moody group (resp. algebra) associated with a generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$. Let $B^\pm$ be fixed Borel subgroups and T the maximal torus such that $B^+ \cap B^- = T$. Set $y_i(c) := \exp(cf_i)$, and let $\alpha_i^\vee(c) \in T$ be the image of $c \in \mathbb{C}^\times$ by the group morphism $\mathbb{C}^\times \to T$ induced by the simple coroot $\alpha_i^\vee$ as in 2.1. We set $Y_i(c) := y_i(c^{-1})\, \alpha_i^\vee(c) = \alpha_i^\vee(c)\, y_i(c)$. Let W (resp. $\widetilde{W}$) be the Weyl group (resp. the extended Weyl group) associated with $\mathfrak{g}$. The Schubert cell $X_w := BwB/B$ $(w = s_{i_1} \cdots s_{i_k} \in W)$ is birationally isomorphic to the variety

$$B_w^- := \left\{Y_{i_1}(x_1) \cdots Y_{i_k}(x_k) \,|\, x_1, \cdots, x_k \in \mathbb{C}^\times\right\} \subset B^-,$$

and X_w has a natural geometric crystal structure ([1], [13]).

We choose $0 \in I$ as in [2], and let $\{\varpi_i\}_{i\in I\setminus\{0\}}$ be the set of level 0 fundamental weights. Let $W(\varpi_i)$ be the fundamental representation of $U_q(\mathfrak{g})$ with ϖ_i as an extremal weight ([2]). Let us denote its specialization at $q = 1$ by the same notation $W(\varpi_i)$. It is a finite-dimensional $\mathfrak{g}$-module. Let $\mathbb{P}(\varpi_i)$ be the projective space $(W(\varpi_i) \setminus \{0\})/\mathbb{C}^\times$.

For any $i \in I$, define $c_i^\vee := \max(1, \frac{2}{(\alpha_i,\alpha_i)})$. Then the translation $t(c_i^\vee \varpi_i)$ belongs to $\widetilde{W}$ (see [8]). For a subset J of I, let us denote by $\mathfrak{g}_J$ the subalgebra of $\mathfrak{g}$ generated by $\{e_i, f_i\}_{i\in J}$. For an integral weight μ, define $I(\mu) := \left\{j \in I \,|\, \langle \alpha_j^\vee, \mu\rangle \geq 0\right\}$.

Here we state the conjecture given in [8]:

Conjecture 1.1 ([8]). For any $i \in I$, there exist a unique variety X endowed with a positive $\mathfrak{g}$-geometric crystal structure and a rational mapping $\pi \colon X \longrightarrow \mathbb{P}(\varpi_i)$ satisfying the following property:

(i) for an arbitrary extremal vector $u \in W(\varpi_i)_\mu$, writing the translation $t(c_i^\vee \mu)$ as $\iota w \in \widetilde{W}$ with a Dynkin diagram automorphism ι and $w = s_{i_1} \cdots s_{i_k}$, there exists a birational mapping $\xi \colon B_w^- \longrightarrow X$ such that ξ is a morphism of $\mathfrak{g}_{I(\mu)}$-geometric crystals and that the composition $\pi \circ \xi \colon B_w^- \to \mathbb{P}(\varpi_i)$ coincides with $Y_{i_1}(x_1) \cdots Y_{i_k}(x_k) \mapsto Y_{i_1}(x_1) \cdots Y_{i_k}(x_k)\overline{u}$, where $\overline{u}$ is the line including u,

(ii) the ultra-discretization(see Sect.2) of X is isomorphic to the crystal $B_\infty(\varpi_i)$ of the Langlands dual $\mathfrak{g}^L$.

In [8], the cases $i = 1$ and $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)},\ A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}$ have been resolved, that is, certain positive geometric crystal $\mathcal{V}(\mathfrak{g})$ associated with the fundamental representation $W(\varpi_1)$ for the above affine Lie algebras has been constructed and it was shown that the ultra-discretization limit of $\mathcal{V}(\mathfrak{g})$ is isomorphic to the limit of the coherent family of perfect crystals as above for $\mathfrak{g}^L$ the Langlands dual of $\mathfrak{g}$. In [15] for the case $i = 1$ and $\mathfrak{g} = G_2^{(1)}$, a positive geometric crystal $\mathcal{V}$ was constructed. However, the ultra-discretization of the geometric crystal has not been given there, though it was conjectured that the ultra-discretization of $\mathcal{V}$ is isomorphic to B_∞ as in [7].

In this article, we shall describe the structure of the crystal obtained by ultra-discretization process from the geometric crystal $\mathcal{V}$ for $\mathfrak{g} = G_2^{(1)}$ in [15]. Finally, we shall show that the crystal is isomorphic to B_∞ as in [7].

2. Geometric crystals

In this section, we review Kac-Moody groups and geometric crystals following [11], [12], [1]

2.1. Kac-Moody algebras and Kac-Moody groups. Fix a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$ with a finite index set I. Let $(\mathfrak{t}, \{\alpha_i\}_{i\in I}, \{\alpha_i^\vee\}_{i\in I})$ be the associated root data, where $\mathfrak{t}$ is a vector space over $\mathbb{C}$ and $\{\alpha_i\}_{i\in I} \subset \mathfrak{t}^*$ and $\{\alpha_i^\vee\}_{i\in I} \subset \mathfrak{t}$ are linearly independent satisfying $\alpha_j(\alpha_i^\vee) = a_{ij}$.

The Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ associated with A is the Lie algebra over $\mathbb{C}$ generated by $\mathfrak{t}$, the Chevalley generators e_i and f_i $(i \in I)$ with the usual defining relations ([10],[11]). There is the root space decomposition $\mathfrak{g} = \bigoplus_{\alpha\in\mathfrak{t}^*} \mathfrak{g}_\alpha$. Denote the set of roots by $\Delta := \{\alpha \in \mathfrak{t}^* | \alpha \neq 0,\ \mathfrak{g}_\alpha \neq (0)\}$. Set $Q = \sum_i \mathbb{Z}\alpha_i$, $Q_+ = \sum_i \mathbb{Z}_{\geq 0}\alpha_i$, $Q^\vee := \sum_i \mathbb{Z}\alpha_i^\vee$ and $\Delta_+ := \Delta \cap Q_+$. An element of Δ_+ is called a *positive root*. Let $P \subset \mathfrak{t}^*$ be a weight lattice such that $\mathbb{C} \otimes P = \mathfrak{t}^*$, whose element is called a weight.

Define simple reflections $s_i \in \mathrm{Aut}(\mathfrak{t})$ $(i \in I)$ by $s_i(h) := h - \alpha_i(h)\alpha_i^\vee$, which generate the Weyl group W. It induces the action of W on $\mathfrak{t}^*$ by $s_i(\lambda) := \lambda - \lambda(\alpha_i^\vee)\alpha_i$. Set $\Delta^{\mathrm{re}} := \{w(\alpha_i) | w \in W,\ i \in I\}$, whose element is called a real root.

Let $\mathfrak{g}'$ be the derived Lie algebra of $\mathfrak{g}$ and let G be the Kac-Moody group associated with $\mathfrak{g}'$([11]). Let $U_\alpha := \exp \mathfrak{g}_\alpha$ $(\alpha \in \Delta^{\mathrm{re}})$ be the one-parameter subgroup of G. The group G is generated by U_α $(\alpha \in \Delta^{\mathrm{re}})$. Let $U^\pm$ be the subgroup generated by $U_{\pm\alpha}$ $(\alpha \in \Delta_+^{\mathrm{re}} = \Delta^{\mathrm{re}} \cap Q_+)$, *i.e.*, $U^\pm := \langle U_{\pm\alpha} | \alpha \in \Delta_+^{\mathrm{re}} \rangle$.

For any $i \in I$, there exists a unique homomorphism; $\phi_i : SL_2(\mathbb{C}) \to G$ such that

$$\phi_i\left(\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}\right) = c^{\alpha_i^\vee},\ \phi_i\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) = \exp(te_i),\ \phi_i\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = \exp(tf_i).$$

where $c \in \mathbb{C}^\times$ and $t \in \mathbb{C}$. Set $\alpha_i^\vee(c) := c^{\alpha_i^\vee}$, $x_i(t) := \exp(te_i)$, $y_i(t) := \exp(tf_i)$, $G_i := \phi_i(SL_2(\mathbb{C}))$, $T_i := \phi_i(\{\mathrm{diag}(c, c^{-1}) | c \in \mathbb{C}^\vee\})$ and $N_i := N_{G_i}(T_i)$. Let T (resp. N) be the subgroup of G with the Lie algebra $\mathfrak{t}$ (resp. generated by the N_i's), which is called a *maximal torus* in G, and let $B^\pm = U^\pm T$ be the Borel subgroup of G. We have the isomorphism $\phi : W \xrightarrow{\sim} N/T$ defined by $\phi(s_i) = N_i T/T$. An

element $\overline{s}_i := x_i(-1)y_i(1)x_i(-1) = \phi_i\left(\begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix}\right)$ is in $N_G(T)$, which is a representative of $s_i \in W = N_G(T)/T$.

2.2. **Geometric crystals.** Let W be the Weyl group associated with $\mathfrak{g}$. Define $R(w)$ for $w \in W$ by

$$R(w) := \{(i_1, i_2, \cdots, i_l) \in I^l | w = s_{i_1} s_{i_2} \cdots s_{i_l}\},$$

where l is the length of w. Then $R(w)$ is the set of reduced words of w.

Let X be an ind-variety , $\gamma_i : X \to \mathbb{C}$ and $\varepsilon_i : X \longrightarrow \mathbb{C}$ $(i \in I)$ rational functions on X, and $e_i : \mathbb{C}^\times \times X \longrightarrow X$ $((c,x) \mapsto e_i^c(x))$ a rational $\mathbb{C}^\times$-action.

For a word $\mathbf{i} = (i_1, \cdots, i_l) \in R(w)$ $(w \in W)$, set $\alpha^{(j)} := s_{i_l} \cdots s_{i_{j+1}}(\alpha_{i_j})$ $(1 \le j \le l)$ and

$$\begin{array}{rrl} e_{\mathbf{i}}: & T \times X \to & X \\ & (t,x) \mapsto & e_{\mathbf{i}}^t(x) := e_{i_1}^{\alpha^{(1)}(t)} e_{i_2}^{\alpha^{(2)}(t)} \cdots e_{i_l}^{\alpha^{(l)}(t)}(x). \end{array}$$

Definition 2.1. A quadruple $(X, \{e_i\}_{i\in I}, \{\gamma_i,\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ is a G (or $\mathfrak{g}$)-*geometric crystal* if

(i) $\{1\} \times X \subset dom(e_i)$ for any $i \in I$.
(ii) $\gamma_j(e_i^c(x)) = c^{a_{ij}}\gamma_j(x)$.
(iii) $e_{\mathbf{i}} = e_{\mathbf{i}'}$ for any $w \in W$, $\mathbf{i}$. $\mathbf{i}' \in R(w)$.
(iv) $\varepsilon_i(e_i^c(x)) = c^{-1}\varepsilon_i(x)$.

Note that the condition (iii) as above is equivalent to the following so-called *Verma relations*:

$$\begin{array}{ll} e_i^{c_1} e_j^{c_2} = e_j^{c_2} e_i^{c_1} & \text{if } a_{ij} = a_{ji} = 0, \\ e_i^{c_1} e_j^{c_1c_2} e_i^{c_2} = e_j^{c_2} e_i^{c_1c_2} e_j^{c_1} & \text{if } a_{ij} = a_{ji} = -1, \\ e_i^{c_1} e_j^{c_1^2c_2} e_i^{c_1c_2} e_j^{c_2} = e_j^{c_2} e_i^{c_1c_2} e_j^{c_1^2c_2} e_i^{c_1} & \text{if } a_{ij} = -2,\ a_{ji} = -1, \\ e_i^{c_1} e_j^{c_1^3c_2} e_i^{c_1^2c_2} e_j^{c_1^3c_2^2} e_i^{c_1c_2} e_j^{c_2} = e_j^{c_2} e_i^{c_1c_2} e_j^{c_1^3c_2^2} e_i^{c_1^2c_2} e_j^{c_1^3c_2} e_i^{c_1} & \text{if } a_{ij} = -3,\ a_{ji} = -1, \end{array}$$

Note that the last formula is different from the one in [1], [13], [14] which seems to be incorrect. The formula here may be correct.

2.3. **Geometric crystal on Schubert cell.** Let $w \in W$ be a Weyl group element and take a reduced expression $w = s_{i_1} \cdots s_{i_l}$. Let $X := G/B$ be the flag variety, which is an ind-variety and $X_w \subset X$ the Schubert cell associated with w, which has a natural geometric crystal structure ([1],[13]). For $\mathbf{i} := (i_1, \cdots, i_k)$, set

$$B_{\mathbf{i}}^- := \{Y_{\mathbf{i}}(c_1, \cdots, c_k) := Y_{i_1}(c_1) \cdots Y_{i_l}(c_k) \,|\, c_1 \cdots, c_k \in \mathbb{C}^\times\} \subset B^-, \tag{2.1}$$

which has a geometric crystal structure([13]) isomorphic to X_w. The explicit forms of the action e_i^c, the rational function ε_i and γ_i on $B_{\mathbf{i}}^-$ are given by

$$e_i^c(Y_{i_1}(c_1)\cdots Y_{i_l}(c_k)) = Y_{i_1}(\mathcal{C}_1)\cdots Y_{i_l}(\mathcal{C}_k)),$$

where

$$\mathcal{C}_j := c_j \cdot \frac{\displaystyle\sum_{1\le m\le j, i_m=i} \frac{c}{c_1^{a_{i_1,i}}\cdots c_{m-1}^{a_{i_{m-1},i}} c_m} + \sum_{j<m\le k, i_m=i} \frac{1}{c_1^{a_{i_1,i}}\cdots c_{m-1}^{a_{i_{m-1},i}} c_m}}{\displaystyle\sum_{1\le m<j, i_m=i} \frac{c}{c_1^{a_{i_1,i}}\cdots c_{m-1}^{a_{i_{m-1},i}} c_m} + \sum_{j\le m\le k, i_m=i} \frac{1}{c_1^{a_{i_1,i}}\cdots c_{m-1}^{a_{i_{m-1},i}} c_m}}, \tag{2.2}$$

$$\varepsilon_i(Y_{i_1}(c_1)\cdots Y_{i_l}(c_k)) = \sum_{1\le m\le k, i_m=i} \frac{1}{c_1^{a_{i_1,i}}\cdots c_{m-1}^{a_{i_{m-1},i}} c_m}, \tag{2.3}$$

$$\gamma_i(Y_{i_1}(c_1)\cdots Y_{i_l}(c_k)) = c_1^{a_{i_1,i}}\cdots c_k^{a_{i_k,i}}. \tag{2.4}$$

2.4. **Positive structure, Ultra-discretizations and Tropicalizations.** Let us recall the notions of positive structure, ultra-discretization and tropicalization.

The setting below is same as [8]. Let $T = (\mathbb{C}^\times)^l$ be an algebraic torus over $\mathbb{C}$ and $X^*(T) := \mathrm{Hom}(T, \mathbb{C}^\times) \cong \mathbb{Z}^l$ (resp. $X_*(T) := \mathrm{Hom}(\mathbb{C}^\times, T) \cong \mathbb{Z}^l$) be the lattice of characters (resp. co-characters) of T. Set $R := \mathbb{C}(c)$ and define

$$\begin{array}{cccc} v: & R\setminus\{0\} & \longrightarrow & \mathbb{Z} \\ & f(c) & \mapsto & \deg(f(c)), \end{array}$$

where deg is the degree of poles at $c = \infty$. Here note that for $f_1, f_2 \in R\setminus\{0\}$, we have

$$v(f_1 f_2) = v(f_1) + v(f_2), \quad v\left(\frac{f_1}{f_2}\right) = v(f_1) - v(f_2) \tag{2.5}$$

A non-zero rational function on an algebraic torus T is called *positive* if it is written as g/h where g and h are a positive linear combination of characters of T.

Definition 2.2. Let $f\colon T \to T'$ be a rational morphism between two algebraic tori T and T'. We say that f is *positive*, if $\chi\circ f$ is positive for any character $\chi\colon T' \to \mathbb{C}$.

Denote by $\mathrm{Mor}^+(T, T')$ the set of positive rational morphisms from T to T'.

Lemma 2.3 ([1]). For any $f \in \mathrm{Mor}^+(T_1, T_2)$ and $g \in \mathrm{Mor}^+(T_2, T_3)$, the composition $g\circ f$ is well-defined and belongs to $\mathrm{Mor}^+(T_1, T_3)$.

By Lemma 2.3, we can define a category $\mathcal{T}_+$ whose objects are algebraic tori over $\mathbb{C}$ and arrows are positive rational morphisms.

Let $f\colon T \to T'$ be a positive rational morphism of algebraic tori T and T'. We define a map $\widehat{f}\colon X_*(T) \to X_*(T')$ by

$$\langle \chi, \widehat{f}(\xi)\rangle = v(\chi\circ f\circ \xi),$$

where $\chi \in X^*(T')$ and $\xi \in X_*(T)$.

Lemma 2.4 ([1]). For any algebraic tori T_1, T_2, T_3, and positive rational morphisms $f \in \mathrm{Mor}^+(T_1, T_2)$, $g \in \mathrm{Mor}^+(T_2, T_3)$, we have $\widehat{g\circ f} = \widehat{g}\circ\widehat{f}$.

By this lemma, we obtain a functor

$$\begin{array}{cccc} \mathcal{UD}: & \mathcal{T}_+ & \longrightarrow & \mathfrak{Set} \\ & T & \mapsto & X_*(T) \\ & (f: T\to T') & \mapsto & (\widehat{f}: X_*(T) \to X_*(T'))) \end{array}$$

Definition 2.5 ([1])**.** Let $\chi=(X,\{e_i\}_{i\in I},\{\mathrm{wt}_i\}_{i\in I},\{\varepsilon_i\}_{i\in I})$ be a geometric crystal, T' an algebraic torus and $\theta:T'\to X$ a birational isomorphism. The isomorphism θ is called *positive structure* on χ if it satisfies

(i) for any $i\in I$ the rational functions $\gamma_i\circ\theta:T'\to\mathbb{C}$ and $\varepsilon_i\circ\theta:T'\to\mathbb{C}$ are positive.

(ii) For any $i\in I$, the rational morphism $e_{i,\theta}:\mathbb{C}^\times\times T'\to T'$ defined by $e_{i,\theta}(c,t):=\theta^{-1}\circ e_i^c\circ\theta(t)$ is positive.

Let $\theta:T\to X$ be a positive structure on a geometric crystal $\chi=(X,\{e_i\}_{i\in I},\{\mathrm{wt}_i\}_{i\in I},\{\varepsilon_i\}_{i\in I})$. Applying the functor $\mathcal{UD}$ to positive rational morphisms $e_{i,\theta}:\mathbb{C}^\times\times T'\to T'$ and $\gamma\circ\theta:T'\to T$ (the notations are as above), we obtain

$$\begin{aligned}
\tilde{e}_i &:= \mathcal{UD}(e_{i,\theta}):\mathbb{Z}\times X_*(T)\to X_*(T)\\
\mathrm{wt}_i &:= \mathcal{UD}(\gamma_i\circ\theta):X_*(T')\to\mathbb{Z},\\
\varepsilon_i &:= \mathcal{UD}(\varepsilon_i\circ\theta):X_*(T')\to\mathbb{Z}.
\end{aligned}$$

Now, for given positive structure $\theta:T'\to X$ on a geometric crystal $\chi=(X,\{e_i\}_{i\in I},\{\mathrm{wt}_i\}_{i\in I},\{\varepsilon_i\}_{i\in I})$, we associate the quadruple $(X_*(T'),\{\tilde{e}_i\}_{i\in I},\{\mathrm{wt}_i\}_{i\in I},\{\varepsilon_i\}_{i\in I})$ with a free pre-crystal structure (see [1, 2.2]) and denote it by $\mathcal{UD}_{\theta,T'}(\chi)$. We have the following theorem:

Theorem 2.6 ([1][13]). For any geometric crystal $\chi=(X,\{e_i\}_{i\in I},\{\gamma_i\}_{i\in I},\{\varepsilon_i\}_{i\in I})$ and positive structure $\theta:T'\to X$, the associated pre-crystal $\mathcal{UD}_{\theta,T'}(\chi)=(X_*(T'),\{e_i\}_{i\in I},\{\mathrm{wt}_i\}_{i\in I},\{\varepsilon_i\}_{i\in I})$ is a crystal (see [1, 2.2])

Now, let $\mathcal{GC}^+$ be a category whose object is a triplet (χ,T',θ) where $\chi=(X,\{e_i\},\{\gamma_i\},\{\varepsilon_i\})$ is a geometric crystal and $\theta:T'\to X$ is a positive structure on χ, and morphism $f:(\chi_1,T_1',\theta_1)\longrightarrow(\chi_2,T_2',\theta_2)$ is given by a morphism $\varphi:X_1\longrightarrow X_2$ ($\chi_i=(X_i,\cdots)$) such that

$$f:=\theta_2^{-1}\circ\varphi\circ\theta_1:T_1'\longrightarrow T_2',$$

is a positive rational morphism. Let $\mathcal{CR}$ be a category of crystals. Then by the theorem above, we have

Corollary 2.7. $\mathcal{UD}_{\theta,T'}$ as above defines a functor

$$\begin{aligned}
\mathcal{UD} \ : \ & \mathcal{GC}^+\longrightarrow\mathcal{CR},\\
& (\chi,T',\theta)\mapsto X_*(T'),\\
& (f:(\chi_1,T_1',\theta_1)\to(\chi_2,T_2',\theta_2))\mapsto(\widehat{f}:X_*(T_1')\to X_*(T_2')).
\end{aligned}$$

We call the functor $\mathcal{UD}$ *"ultra-discretization"* as [13],[14] instead of "tropicalization" as in [1]. And for a crystal B, if there exists a geometric crystal χ and a positive structure $\theta:T'\to X$ on χ such that $\mathcal{UD}(\chi,T',\theta)\cong B$ as crystals, we call an object (χ,T',θ) in $\mathcal{GC}^+$ a *tropicalization* of B, where it is not known that this correspondence is a functor.

3. Limit of perfect crystals

We review limit of perfect crystals following [4]. (See also [5],[6]).

3.1. **Crystals.** First we review the theory of crystals, which is the notion obtained by abstracting the combinatorial properties of crystal bases. Let $(A, \{\alpha_i\}_{i\in I}, \{\alpha_i^\vee\}_{\imath\in I})$ be a Cartan data.

Definition 3.1. A *crystal* B is a set endowed with the following maps:

$$\begin{aligned}
&\mathrm{wt} : B \longrightarrow P,\\
&\varepsilon_i : B \longrightarrow \mathbb{Z}\sqcup\{-\infty\}, \quad \varphi_i : B \longrightarrow \mathbb{Z}\sqcup\{-\infty\} \quad \text{for} \quad i\in I,\\
&\tilde{e}_i : B\sqcup\{0\} \longrightarrow B\sqcup\{0\}, \quad \tilde{f}_i : B\sqcup\{0\} \longrightarrow B\sqcup\{0\} \quad \text{for} \quad i\in I,\\
&\tilde{e}_i(0) = \tilde{f}_i(0) = 0.
\end{aligned}$$

those maps satisfy the following axioms: for all $b, b_1, b_2 \in B$, we have

$$\begin{aligned}
&\varphi_i(b) = \varepsilon_i(b) + \langle\alpha_i^\vee, \mathrm{wt}(b)\rangle,\\
&\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i \text{ if } \tilde{e}_i b \in B,\\
&\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i \text{ if } \tilde{f}_i b \in B,\\
&\tilde{e}_i b_2 = b_1 \Longleftrightarrow \tilde{f}_i b_1 = b_2 \ (b_1, b_2 \in B),\\
&\varepsilon_i(b) = -\infty \Longrightarrow \tilde{e}_i b = \tilde{f}_i b = 0.
\end{aligned}$$

The following tensor product structure is one of the most crucial properties of crystals.

Theorem 3.2. Let B_1 and B_2 be crystals. Set $B_1\otimes B_2 := \{b_1\otimes b_2;\ b_j \in B_j\ (j = 1,2)\}$. Then we have

(i) $B_1 \otimes B_2$ is a crystal.
(ii) For $b_1 \in B_1$ and $b_2 \in B_2$, we have

$$\tilde{f}_i(b_1\otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2),\\ b_1\otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \le \varepsilon_i(b_2).\end{cases}$$

$$\tilde{e}_i(b_1\otimes b_2) = \begin{cases} b_1\otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2),\\ \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \ge \varepsilon_i(b_2),\end{cases}$$

Definition 3.3. Let B_1 and B_2 be crystals. A *strict morphism* of crystals $\psi : B_1 \longrightarrow B_2$ is a map $\psi : B_1\sqcup\{0\} \longrightarrow B_2\sqcup\{0\}$ satisfying: $\psi(0) = 0$, $\psi(B_1)\subset B_2$, ψ commutes with all $\tilde{e}_i$ and $\tilde{f}_i$ and

$$\mathrm{wt}(\psi(b)) = \mathrm{wt}(b), \quad \varepsilon_i(\psi(b)) = \varepsilon_i(b), \quad \varphi_i(\psi(b)) = \varphi_i(b) \text{ for any } b\in B_1.$$

In particular, a bijective strict morphism is called an *isomorphism of crystals*.

Example 3.4. If (L, B) is a crystal base, then B is a crystal. Hence, for the crystal base $(L(\infty), B(\infty))$ of the nilpotent subalgebra $U_q^-(\mathfrak{g})$ of the quantum algebra $U_q(\mathfrak{g})$, $B(\infty)$ is a crystal.

Example 3.5. For $\lambda\in P$, set $T_\lambda := \{t_\lambda\}$. We define a crystal structure on T_λ by

$$\tilde{e}_i(t_\lambda) = \tilde{f}_i(t_\lambda) = 0, \quad \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty, \quad \mathrm{wt}(t_\lambda) = \lambda.$$

Definition 3.6. For a crystal B, a colored oriented graph structure is associated with B by

$$b_1 \xrightarrow{i} b_2 \Longleftrightarrow \tilde{f}_i b_1 = b_2.$$

We call this graph a *crystal graph* of B.

3.2. **Affine weights.** Let $\mathfrak{g}$ be an affine Lie algebra. The sets $\mathfrak{t}$, $\{\alpha_i\}_{i\in I}$ and $\{\alpha_i^\vee\}_{i\in I}$ be as in 2.1. We take $\dim\mathfrak{t} = \sharp I + 1$. Let $\delta \in Q_+$ be the unique element satisfying $\{\lambda \in Q | \langle \alpha_i^\vee, \lambda\rangle = 0 \text{ for any } i \in I\} = \mathbb{Z}\delta$ and $\mathbf{c} \in \mathfrak{g}$ be the canonical central element satisfying $\{h \in Q^\vee | \langle h, \alpha_i\rangle = 0 \text{ for any } i \in I\} = \mathbb{Z}c$. We write ([9, 6.1])

$$\mathbf{c} = \sum_i a_i^\vee \alpha_i^\vee, \qquad \delta = \sum_i a_i \alpha_i.$$

Let $(\quad,\quad)$ be the non-degenerate W-invariant symmetric bilinear form on $\mathfrak{t}^*$ normalized by $(\delta, \lambda) = \langle \mathbf{c}, \lambda\rangle$ for $\lambda \in \mathfrak{t}^*$. Let us set $\mathfrak{t}^*_{\rm cl} := \mathfrak{t}^*/\mathbb{C}\delta$ and let ${\rm cl} : \mathfrak{t}^* \longrightarrow \mathfrak{t}^*_{\rm cl}$ be the canonical projection. Here we have $\mathfrak{t}^*_{\rm cl} \cong \oplus_i (\mathbb{C}\alpha_i^\vee)^*$. Set $\mathfrak{t}^*_0 := \{\lambda \in \mathfrak{t}^* | \langle \mathbf{c}, \lambda\rangle = 0\}$, $(\mathfrak{t}^*_{\rm cl})_0 := {\rm cl}(\mathfrak{t}^*_0)$. Since $(\delta, \delta) = 0$, we have a positive-definite symmetric form on $\mathfrak{t}^*_{\rm cl}$ induced by the one on $\mathfrak{t}^*$. Let $\Lambda_i \in \mathfrak{t}^*_{\rm cl}$ $(i \in I)$ be a classical weight such that $\langle \alpha_i^\vee, \Lambda_j\rangle = \delta_{i,j}$, which is called a fundamental weight. We choose P so that $P_{\rm cl} := {\rm cl}(P)$ coincides with $\oplus_{i\in I}\mathbb{Z}\Lambda_i$ and we call $P_{\rm cl}$ a *classical weight lattice*.

3.3. **Definitions of perfect crystal and its limit.** Let $\mathfrak{g}$ be an affine Lie algebra, P_{cl} be a classical weight lattice as above and set $(P_{cl})^+_l := \{\lambda \in P_{cl} | \langle c, \lambda\rangle = l,\ \langle \alpha_i^\vee, \lambda\rangle \geq 0\}$ $(l \in \mathbb{Z}_{>0})$.

Definition 3.7. A crystal B is a *perfect* of level l if

(i) $B \otimes B$ is connected as a crystal graph.
(ii) There exists $\lambda_0 \in P_{\rm cl}$ such that

$$\mathrm{wt}(B) \subset \lambda_0 + \sum_{i\neq 0} \mathbb{Z}_{\leq 0}{\rm cl}(\alpha_i), \qquad \sharp B_{\lambda_0} = 1$$

(iii) There exists a finite-dimensional $U_q'(\mathfrak{g})$-module V with a crystal pseudo-base B_{ps} such that $B \cong B_{ps}/\pm 1$
(iv) The maps $\varepsilon, \varphi : B^{min} := \{b \in B | \langle c, \varepsilon(b)\rangle = l\} \longrightarrow (P^+_{\rm cl})_l$ are bijective, where $\varepsilon(b) := \sum_i \varepsilon_i(b)\Lambda_i$ and $\varphi(b) := \sum_i \varphi_i(b)\Lambda_i$.

Let $\{B_l\}_{l\geq 1}$ be a family of perfect crystals of level l and set $J := \{(l, b) | l > 0,\ b \in B_l^{min}\}$.

Definition 3.8. A crystal B_∞ with an element b_∞ is called a *limit of* $\{B_l\}_{l\geq 1}$ if

(i) $\mathrm{wt}(b_\infty) = \varepsilon(b_\infty) = \varphi(b_\infty) = 0$.
(ii) For any $(l, b) \in J$, there exists an embedding of crystals:

$$\begin{aligned} f_{(l,b)} : \quad & T_{\varepsilon(b)} \otimes B_l \otimes T_{-\varphi(b)} \hookrightarrow B_\infty \\ & t_{\varepsilon(b)} \otimes b \otimes t_{-\varphi(b)} \mapsto b_\infty \end{aligned}$$

(iii) $B_\infty = \bigcup_{(l,b)\in J} {\rm Im} f_{(l,b)}$.

As for the crystal T_λ, see Example 3.5. If a limit exists for a family $\{B_l\}$, we say that $\{B_l\}$ is a *coherent family* of perfect crystals.

The following is one of the most important properties of limit of perfect crystals.

Proposition 3.9. Let $B(\infty)$ be the crystal as in Example 3.4. Then we have the following isomorphism of crystals:

$$B(\infty) \otimes B_\infty \xrightarrow{\sim} B(\infty).$$

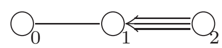
4. Perfect Crystals of type $D_4^{(3)}$

In this section, we review the family of perfect crystals of type $D_4^{(3)}$ and its limit([7]).

We fix the data for $D_4^{(3)}$. Let $\{\alpha_0, \alpha_1, \alpha_2\}$, $\{\alpha_0^\vee, \alpha_1^\vee, \alpha_2^\vee\}$ and $\{\Lambda_0, \Lambda_1, \Lambda_2\}$ be the set of simple roots, simple coroots and fundamental weights, respectively. The Cartan matrix $A = (a_{ij})_{i,j=0,1,2}$ is given by

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix},$$

and its Dynkin diagram is as follows.



The standard null root δ and the canonical central element c are given by

$$\delta = \alpha_0 + 2\alpha_1 + \alpha_2 \quad \text{and} \quad c = \alpha_0^\vee + 2\alpha_1^\vee + 3\alpha_2^\vee,$$

where $\alpha_0 = 2\Lambda_0 - \Lambda_1 + \delta, \quad \alpha_1 = -\Lambda_0 + 2\Lambda_1 - \Lambda_2, \quad \alpha_2 = -3\Lambda_1 + 2\Lambda_2.$

For a positive integer l we introduce $D_4^{(3)}$-crystals B_l and B_∞ as

$$B_l = \left\{ b = (b_1, b_2, b_3, \bar{b}_3, \bar{b}_2, \bar{b}_1) \in (\mathbb{Z}_{\geq 0})^6 \;\middle|\; \begin{array}{l} b_3 \equiv \bar{b}_3 \pmod 2, \\ \sum_{i=1,2}(b_i + \bar{b}_i) + \frac{b_3 + \bar{b}_3}{2} \leq l \end{array} \right\},$$

$$B_\infty = \left\{ b = (b_1, b_2, b_3, \bar{b}_3, \bar{b}_2, \bar{b}_1) \in (\mathbb{Z})^6 \;\middle|\; \begin{array}{l} b_3 \equiv \bar{b}_3 \pmod 2, \\ \sum_{i=1,2}(b_i + \bar{b}_i) + \frac{b_3 + \bar{b}_3}{2} \in \mathbb{Z} \end{array} \right\}.$$

Now we describe the explicit crystal structures of B_l and B_∞. Indeed, most of them coincide with each other except for ε_0 and φ_0. In the rest of this section, we use the following convention: $(x)_+ = \max(x, 0)$.

$$\tilde{e}_1 b = \begin{cases} (\ldots, \bar{b}_2 + 1, \bar{b}_1 - 1) & \text{if } \bar{b}_2 - \bar{b}_3 \geq (b_2 - b_3)_+, \\ (\ldots, b_3 + 1, \bar{b}_3 - 1, \ldots) & \text{if } \bar{b}_2 - \bar{b}_3 < 0 \leq b_3 - b_2, \\ (b_1 + 1, b_2 - 1, \ldots) & \text{if } (\bar{b}_2 - \bar{b}_3)_+ < b_2 - b_3, \end{cases}$$

$$\tilde{f}_1 b = \begin{cases} (b_1 - 1, b_2 + 1, \ldots) & \text{if } (\bar{b}_2 - \bar{b}_3)_+ \leq b_2 - b_3, \\ (\ldots, b_3 - 1, \bar{b}_3 + 1, \ldots) & \text{if } \bar{b}_2 - \bar{b}_3 \leq 0 < b_3 - b_2, \\ (\ldots, \bar{b}_2 - 1, \bar{b}_1 + 1) & \text{if } \bar{b}_2 - \bar{b}_3 > (b_2 - b_3)_+, \end{cases}$$

$$\tilde{e}_2 b = \begin{cases} (\ldots, \bar{b}_3 + 2, \bar{b}_2 - 1, \ldots) & \text{if } \bar{b}_3 \geq b_3, \\ (\ldots, b_2 + 1, b_3 - 2, \ldots) & \text{if } \bar{b}_3 < b_3, \end{cases}$$

$$\tilde{f}_2 b = \begin{cases} (\ldots, b_2 - 1, b_3 + 2, \ldots) & \text{if } \bar{b}_3 \leq b_3, \\ (\ldots, \bar{b}_3 - 2, \bar{b}_2 + 1, \ldots) & \text{if } \bar{b}_3 > b_3, \end{cases}$$

$$\varepsilon_1(b) = \bar{b}_1 + (\bar{b}_3 - \bar{b}_2 + (b_2 - b_3)_+)_+, \qquad \varphi_1(b) = b_1 + (b_3 - b_2 + (\bar{b}_2 - \bar{b}_3)_+)_+,$$
$$\varepsilon_2(b) = \bar{b}_2 + \frac{1}{2}(b_3 - \bar{b}_3)_+, \qquad \varphi_2(b) = b_2 + \frac{1}{2}(\bar{b}_3 - b_3)_+,$$
$$\varepsilon_0(b) = \begin{cases} l - s(b) + \max A - (2z_1 + z_2 + z_3 + 3z_4) & b \in B_l, \\ -s(b) + \max A - (2z_1 + z_2 + z_3 + 3z_4) & b \in B\infty. \end{cases}$$
$$\varphi_0(b) = \begin{cases} l - s(b) + \max A & b \in B_l, \\ -s(b) + \max A & b \in B_\infty, \end{cases}$$

where

$$s(b) = b_1 + b_2 + \frac{b_3 + \bar{b}_3}{2} + \bar{b}_2 + \bar{b}_1. \tag{4.1}$$

$$z_1 = \bar{b}_1 - b_1, \quad z_2 = \bar{b}_2 - \bar{b}_3, \quad z_3 = b_3 - b_2, \quad z_4 = (\bar{b}_3 - b_3)/2, \tag{4.2}$$

$$A = (0, z_1, z_1 + z_2, z_1 + z_2 + 3z_4, z_1 + z_2 + z_3 + 3z_4, 2z_1 + z_2 + z_3 + 3z_4) \tag{4.3}$$

For $b \in B_l$ if $\tilde{e}_i b$ or $\tilde{f}_i b$ does not belong to B_l, namely, if b_j or $\bar{b}_j$ for some j becomes negative, we understand it to be 0.

Let us see the actions of $\tilde{e}_0$ and $\tilde{f}_0$. We shall consider the conditions (E_1)-(E_6) and (F_1)-(F_6) ([7]).

$$\begin{array}{ll}
(E_1) & z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + z_2 + 3z_4 < 0, z_1 + z_2 < 0, z_1 < 0, \\
(E_2) & z_1 + z_2 + z_3 + 3z_4 < 0, z_2 + 3z_4 < 0, z_2 < 0, z_1 \geq 0, \\
(E_3) & z_1 + z_3 + 3z_4 < 0, z_3 + 3z_4 < 0, z_4 < 0, z_2 \geq 0, z_1 + z_2 \geq 0, \\
(E_4) & z_1 + z_2 + 3z_4 \geq 0, z_2 + 3z_4 \geq 0, z_4 \geq 0, z_3 < 0, z_1 + z_3 < 0, \\
(E_5) & z_1 + z_2 + z_3 + 3z_4 \geq 0, z_3 + 3z_4 \geq 0, z_3 \geq 0, z_1 < 0, \\
(E_6) & z_1 + z_2 + z_3 + 3z_4 \geq 0, z_1 + z_3 + 3z_4 \geq 0, z_1 + z_3 \geq 0, z_1 \geq 0.
\end{array}$$

(F_i) $(1 \leq i \leq 6)$ is obtained from (E_i) by replacing $\geq$ (resp. $<$) with $>$ (resp. $\leq$). We define

$$\tilde{e}_0 b = \begin{cases} \mathcal{E}_1 b := (b_1 - 1, \ldots) & \text{if } (E_1), \\ \mathcal{E}_2 b := (\ldots, b_3 - 1, \bar{b}_3 - 1, \ldots, \bar{b}_1 + 1) & \text{if } (E_2), \\ \mathcal{E}_3 b := (\ldots, b_3 - 2, \ldots, \bar{b}_2 + 1, \ldots) & \text{if } (E_3), \\ \mathcal{E}_4 b := (\ldots, b_2 - 1, \ldots, \bar{b}_3 + 2, \ldots) & \text{if } (E_4), \\ \mathcal{E}_5 b := (b_1 - 1, \ldots, b_3 + 1, \bar{b}_3 + 1, \ldots) & \text{if } (E_5), \\ \mathcal{E}_6 b := (\ldots, \bar{b}_1 + 1) & \text{if } (E_6), \end{cases}$$

$$\tilde{f}_0 b = \begin{cases} \mathcal{F}_1 b := (b_1 + 1, \ldots) & \text{if } (F_1), \\ \mathcal{F}_2 b := (\ldots, b_3 + 1, \bar{b}_3 + 1, \ldots, \bar{b}_1 - 1) & \text{if } (F_2), \\ \mathcal{F}_3 b := (\ldots, b_3 + 2, \ldots, \bar{b}_2 - 1, \ldots) & \text{if } (F_3), \\ \mathcal{F}_4 b := (\ldots, b_2 + 1, \ldots, \bar{b}_3 - 2, \ldots) & \text{if } (F_4), \\ \mathcal{F}_5 b := (b_1 + 1, \ldots, b_3 - 1, \bar{b}_3 - 1, \ldots) & \text{if } (F_5), \\ \mathcal{F}_6 b := (\ldots, \bar{b}_1 - 1) & \text{if } (F_6). \end{cases}$$

The following is one of the main results in [7]:

Theorem 4.1 ([7]). (i) The $D_4^{(3)}$-crystal B_l is a perfect crystal of level l.

(ii) The family of the perfect crystals $\{B_l\}_{l\geq 1}$ forms a coherent family and the crystal B_∞ is its limit with the vector $b_\infty = (0,0,0,0,0,0)$.

As was shown in [7], the minimal elements are given

$$(B_l)_{\min} = \{(\alpha,\beta,\beta,\beta,\beta,\alpha) \mid \alpha,\beta \in \mathbb{Z}_{\geq 0}, 2\alpha + 3\beta \leq l\}.$$

Let $J = \{(l,b) \mid l \in \mathbb{Z}_{\geq 1}, b \in (B_l)_{\min}\}$ and the maps ε, $\varphi : (B_l)_{\min} \to (P_{\mathrm{cl}}^+)_l$ be as in Sect.3. Then we have $\mathrm{wt} b_\infty = 0$ and $\varepsilon_i(b_\infty) = \varphi_i(b_\infty) = 0$ for $i = 0,1,2$.

For $(l,b_0) \in J$, since $\varepsilon(b_0) = \varphi(b_0)$, one can set $\lambda = \varepsilon(b_0) = \varphi(b_0)$. For $b = (b_1,b_2,b_3,\bar{b}_3,\bar{b}_2,\bar{b}_1) \in B_l$ we define a map

$$f_{(l,b_0)} : T_\lambda \otimes B_l \otimes B_{-\lambda} \longrightarrow B_\infty$$

by

$$f_{(l,b_0)}(t_\lambda \otimes b \otimes t_{-\lambda}) = b' = (\nu_1,\nu_2,\nu_3,\bar{\nu}_3,\bar{\nu}_2,\bar{\nu}_1)$$

where $b_0 = (\alpha,\beta,\beta,\beta,\beta,\alpha)$, and

$$\nu_1 = b_1 - \alpha, \qquad \bar{\nu}_1 = \bar{b}_1 - \alpha,$$
$$\nu_j = b_j - \beta, \qquad \bar{\nu}_j = \bar{b}_j - \beta \ (j = 2,3).$$

Finally, we obtain $B_\infty = \bigcup_{(l,b)\in J} \mathrm{Im}\, f_{(l,b)}$

5. Fundamental Representation for $G_2^{(1)}$

5.1. **Fundamental representation $W(\varpi_1)$.** Let $c = \sum_i a_i^\vee \alpha_i^\vee$ be the canonical central element in an affine Lie algebra $\mathfrak{g}$ (see [9, 6.1]), $\{\Lambda_i | i \in I\}$ the set of fundamental weight as in the previous section and $\varpi_1 := \Lambda_1 - a_1^\vee \Lambda_0$ the (level 0)fundamental weight. Let $W(\varpi_1)$ be the fundamental representation of $U_q'(\mathfrak{g})$ associated with ϖ_1 ([2]).

By [2, Theorem 5.17], $W(\varpi_1)$ is a finite-dimensional irreducible integrable $U_q'(\mathfrak{g})$-module and has a global basis with a simple crystal. Thus, we can consider the specialization $q = 1$ and obtain the finite-dimensional $\mathfrak{g}$-module $W(\varpi_1)$, which we call a fundamental representation of $\mathfrak{g}$ and use the same notation as above.

We shall present the explicit form of $W(\varpi_1)$ for $\mathfrak{g} = G_2^{(1)}$.

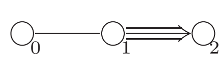
5.2. **$W(\varpi_1)$ for $G_2^{(1)}$.** The Cartan matrix $A = (a_{i,j})_{i,j=0,1,2}$ of type $G_2^{(1)}$ is:

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}.$$

Then the simple roots are

$$\alpha_0 = 2\Lambda_0 - \Lambda_1 + \delta, \quad \alpha_1 = -\Lambda_0 + 2\Lambda_1 - 3\Lambda_2, \quad \alpha_2 = -\Lambda_1 + 2\Lambda_2,$$

and the Dynkin diagram is:



The $\mathfrak{g}$-module $W(\varpi_1)$ is a 15 dimensional module with the basis,

$$\{\boxed{i}, \boxed{\bar{i}}, \emptyset, \boxed{0_1}, \boxed{0_2} \mid i = 1,\cdots,6\}.$$

The following description of $W(\varpi_1)$ slightly differs from [16].

$\mathrm{wt}(\boxed{1}) = \Lambda_1 - 2\Lambda_0,\ \mathrm{wt}(\boxed{2}) = -\Lambda_0 - \Lambda_1 + 3\Lambda_2,\ \mathrm{wt}(\boxed{3}) = -\Lambda_0 + \Lambda_2,$

$\mathrm{wt}(\boxed{4}) = -\Lambda_0 + \Lambda_1 - \Lambda_2,\ \mathrm{wt}(\boxed{5}) = -\Lambda_1 + 2\Lambda_2,\ \mathrm{wt}(\boxed{6}) = -\Lambda_0 + 2\Lambda_1 - 3\Lambda_2,$

$\mathrm{wt}(\boxed{\bar{i}}) = -\mathrm{wt}(\boxed{i})\ (i = 1, \cdots, 6),\ \mathrm{wt}(\boxed{0_1}) = \mathrm{wt}(\boxed{0_2}) = \mathrm{wt}(\emptyset) = 0.$

The actions of e_i and f_i on these basis vectors are given as follows:

$$f_0\left(\boxed{0_2}, \boxed{\bar{6}}, \boxed{\bar{4}}, \boxed{\bar{3}}, \boxed{\bar{2}}, \boxed{\bar{1}}, \emptyset\right) = \left(\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}, \boxed{6}, \emptyset, 2\boxed{1}\right),$$

$$e_0\left(\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}, \boxed{6}, \boxed{0_2}, \emptyset\right) = \left(\emptyset, \boxed{\bar{6}}, \boxed{\bar{4}}, \boxed{\bar{3}}, \boxed{\bar{2}}, \boxed{\bar{1}}, 2\boxed{\bar{1}}\right),$$

$$f_1\left(\boxed{1}, \boxed{4}, \boxed{6}, \boxed{0_1}, \boxed{0_2}, \boxed{\bar{5}}, \boxed{\bar{2}}, \emptyset\right) = \left(\boxed{2}, \boxed{5}, \boxed{0_2}, 3\boxed{\bar{6}}, 2\boxed{\bar{6}}, \boxed{\bar{4}}, \boxed{\bar{1}}, \boxed{\bar{6}}\right),$$

$$e_1\left(\boxed{2}, \boxed{5}, \boxed{0_1}, \boxed{0_2}, \boxed{\bar{6}}, \boxed{\bar{4}}, \boxed{\bar{1}}, \emptyset\right) = \left(\boxed{1}, \boxed{4}, 3\boxed{6}, 2\boxed{6}, \boxed{0_2}, \boxed{\bar{5}}, \boxed{\bar{2}}, \boxed{6}\right),$$

$$f_2\left(\boxed{2}, \boxed{3}, \boxed{4}, \boxed{5}, \boxed{0_1}, \boxed{0_2}, \boxed{\bar{6}}, \boxed{\bar{4}}, \boxed{\bar{3}}\right)$$
$$= \left(\boxed{3}, 2\boxed{4}, 3\boxed{6}, \boxed{0_1}, 2\boxed{\bar{5}}, \boxed{\bar{5}}, \boxed{\bar{4}}, 2\boxed{\bar{3}}, 3\boxed{\bar{2}}\right),$$

$$e_2\left(\boxed{3}, \boxed{4}, \boxed{6}, \boxed{0_1}, \boxed{0_2}, \boxed{\bar{5}}, \boxed{\bar{4}}, \boxed{\bar{3}}, \boxed{\bar{2}}\right)$$
$$= \left(3\boxed{2}, 2\boxed{3}, \boxed{4}, 2\boxed{5}, \boxed{5}, \boxed{0_1}, 3\boxed{\bar{6}}, 2\boxed{\bar{4}}, \boxed{\bar{3}}\right),$$

where we give non-trivial actions only.

6. Affine Geometric Crystal $\mathcal{V}_1(G_2^{(1)})$

Let us review the construction of the affine geometric crystal $\mathcal{V}(G_2^{(1)})$ in $W(\varpi_1)$ following [15].

For $\xi \in (\mathfrak{t}^*_{\mathrm{cl}})_0$, let $t(\xi)$ be the shift as in [2, Sect 4]. Then we have

$$t(\widetilde{\varpi}_1) = s_0 s_1 s_2 s_1 s_2 s_1 =: w_1,$$
$$t(\mathrm{wt}(\boxed{\bar{2}})) = s_2 s_1 s_2 s_1 s_0 s_1 =: w_2,$$

Associated with these Weyl group elements w_1 and w_2, we define algebraic varieties $\mathcal{V}_1 = \mathcal{V}_1(G_2^{(1)})$ and $\mathcal{V}_2 = \mathcal{V}_2(G_2^{(1)}) \subset W(\varpi_1)$ respectively:

$$\mathcal{V}_1 := \{v_1(x) := Y_0(x_0)Y_1(x_1)Y_2(x_2)Y_1(x_3)Y_2(x_4)Y_1(x_5)\boxed{1} \mid x_i \in \mathbb{C}^\times, (0 \le i \le 5)\},$$
$$\mathcal{V}_2 := \{v_2(y) := Y_2(y_2)Y_1(y_1)Y_2(y_4)Y_1(y_3)Y_0(y_0)Y_1(y_5)\boxed{\bar{2}} \mid y_i \in \mathbb{C}^\times, (0 \le i \le 5)\}.$$

Owing to the explicit forms of f_i's on $W(\varpi_1)$ as above, we have $f_0^3 = 0$, $f_1^3 = 0$ and $f_2^4 = 0$ and then

$$Y_i(c) = (1 + \frac{f_i}{c} + \frac{f_i^2}{2c^2})\alpha_i^\vee(c)\ (i = 0, 1), \quad Y_2(c) = (1 + \frac{f_2}{c} + \frac{f_2^2}{2c^2} + \frac{f_2^3}{6c^3})\alpha_2^\vee(c).$$

We get explicit forms of $v_1(x) \in \mathcal{V}_1$ and $v_2(y) \in \mathcal{V}_2$ as in [15]:

$$v_1(x) = \sum_{1 \le i \le 6} \left(X_i\boxed{i} + X_{\bar{i}}\boxed{\bar{i}}\right) + X_{0_1}\boxed{0_1} + X_{0_2}\boxed{0_2} + X_\emptyset \emptyset,$$

$$v_2(y) = \sum_{1 \le i \le 6} \left(Y_i\boxed{i} + Y_{\bar{i}}\boxed{\bar{i}}\right) + Y_{0_1}\boxed{0_1} + Y_{0_2}\boxed{0_2} + Y_\emptyset \emptyset.$$

where the rational functions X_i's and Y_i's are all positive (as for their explicit forms, see [15]) and then we get the positive birational isomorphism $\overline{\sigma}:\mathcal{V}_1\longrightarrow\mathcal{V}_2$ ($v_1(x)\mapsto v_2(y)$) and its inverse $\overline{\sigma}^{-1}$ is also positive. The actions of e_0^c on $v_2(y)$ (respectively $\gamma_0(v_2(y))$ and $\varepsilon_0(v_2(y))$) are induced from the ones on $Y_2(y_2)Y_1(y_1)Y_2(y_4)Y_1(y_3)Y_0(y_0)Y_1(y_5)$ as an element of the geometric crystal $\mathcal{V}_2$. We define the action e_0^c on $v_1(x)$ by

$$e_0^c v_1(x)=\overline{\sigma}^{-1}\circ e_0^c\circ\overline{\sigma}(v_1(x))). \tag{6.1}$$

We also define $\gamma_0(v_1(x))$ and $\varepsilon_0(v_1(x))$ by

$$\gamma_0(v_1(x))=\gamma_0(\overline{\sigma}(v_1(x))),\qquad \varepsilon_0(v_1(x)):=\varepsilon_0(\overline{\sigma}(v_1(x))). \tag{6.2}$$

Theorem 6.1 ([15]). Together with (6.1), (6.2) on $\mathcal{V}_1$, we obtain a positive affine geometric crystal $\chi:=(\mathcal{V}_1,\{e_i\}_{i\in I},\{\gamma_i\}_{i\in I},\{\varepsilon_i\}_{i\in I})$ $(I=\{0,1,2\})$, whose explicit form is as follows: first we have e_i^c, γ_i and ε_i for $i=1,2$ from the formula (2.2), (2.3) and (2.4).

$$e_1^c(v_1(x))=v_1(x_0,\mathcal{C}_1x_1,x_2,\mathcal{C}_3x_3,x_4,\mathcal{C}_5x_5),\ e_2^c(v_1(x))=v_1(x_0,x_1,\mathcal{C}_2x_2,x_3,\mathcal{C}_4x_4,x_5),$$

where

$$\mathcal{C}_1=\frac{\frac{c\,x_0}{x_1}+\frac{x_0\,{x_2}^3}{{x_1}^2\,x_3}+\frac{x_0\,{x_2}^3\,{x_4}^3}{{x_1}^2\,{x_3}^2\,x_5}}{\frac{x_0}{x_1}+\frac{x_0\,{x_2}^3}{{x_1}^2\,x_3}+\frac{x_0\,{x_2}^3\,{x_4}^3}{{x_1}^2\,{x_3}^2\,x_5}},\quad \mathcal{C}_3=\frac{\frac{c\,x_0}{x_1}+\frac{c\,x_0\,{x_2}^3}{{x_1}^2\,x_3}+\frac{x_0\,{x_2}^3\,{x_4}^3}{{x_1}^2\,{x_3}^2\,x_5}}{\frac{c\,x_0}{x_1}+\frac{x_0\,{x_2}^3}{{x_1}^2\,x_3}+\frac{x_0\,{x_2}^3\,{x_4}^3}{{x_1}^2\,{x_3}^2\,x_5}},$$

$$\mathcal{C}_5=\frac{c\left(\frac{x_0}{x_1}+\frac{x_0\,{x_2}^3}{{x_1}^2\,x_3}+\frac{x_0\,{x_2}^3\,{x_4}^3}{{x_1}^2\,{x_3}^2\,x_5}\right)}{\frac{c\,x_0}{x_1}+\frac{c\,x_0\,{x_2}^3}{{x_1}^2\,x_3}+\frac{x_0\,{x_2}^3\,{x_4}^3}{{x_1}^2\,{x_3}^2\,x_5}},\ \mathcal{C}_2=\frac{\frac{c\,x_1}{x_2}+\frac{x_1\,x_3}{{x_2}^2\,x_4}}{\frac{x_1}{x_2}+\frac{x_1\,x_3}{{x_2}^2\,x_4}},\ \mathcal{C}_4=\frac{c\left(\frac{x_1}{x_2}+\frac{x_1\,x_3}{{x_2}^2\,x_4}\right)}{\frac{c\,x_1}{x_2}+\frac{x_1\,x_3}{{x_2}^2\,x_4}},$$

$$\varepsilon_1(v_1(x))=\frac{x_0}{x_1}+\frac{x_0\,{x_2}^3}{{x_1}^2\,x_3}+\frac{x_0\,{x_2}^3\,{x_4}^3}{{x_1}^2\,{x_3}^2\,x_5},\quad \varepsilon_2(v_1(x))=\frac{x_1}{x_2}+\frac{x_1\,x_3}{{x_2}^2\,x_4},$$

$$\gamma_1(v_1(x))=\frac{x_1^2x_3^2x_5^2}{x_0x_2^3x_4^3},\quad \gamma_2(v_1(x))=\frac{x_2^2x_4^2}{x_1x_3x_5}.$$

We also have e_0^c, ε_0 and γ_0 on $v_1(x)$ as:

$$e_0^c(v_1(x))=v_1\Big(\frac{D}{c\cdot E}x_0,\frac{F}{c\cdot E}x_1,\frac{G}{c\cdot E}x_2,\frac{D\cdot H}{c^2\cdot E\cdot F}x_3,\frac{D}{c\cdot G}x_4,\frac{D}{c\cdot H}x_5\Big),$$

$$\varepsilon_0(v_1(x))=\frac{E}{{x_0}^3\,{x_2}^3\,x_3},\qquad \gamma_0(v_1(x))=\frac{x_0^2}{x_1x_3x_5},$$

where

$$\begin{aligned}
D&=c^2\,{x_0}^2\,{x_2}^3\,x_3+x_1\,{x_2}^3\,{x_3}^2\,x_5+c\,x_0\,(x_1\,{x_3}^3+3\,x_1\,x_2\,{x_3}^2\,x_4\\
&\qquad+3\,x_1\,{x_2}^2\,x_3\,{x_4}^2+{x_2}^3\,\left({x_3}^2+x_1\,{x_4}^3+x_1\,x_3\,x_5\right)),\\
E&={x_0}^2\,{x_2}^3\,x_3+x_1\,{x_2}^3\,{x_3}^2\,x_5+x_0\,\left(x_1\,{x_3}^3+3\,x_1\,x_2\,{x_3}^2\,x_4+3\,x_1\,{x_2}^2\,x_3\,{x_4}^2\right.\\
&\qquad\left.+{x_2}^3\,\left({x_3}^2+x_1\,{x_4}^3+x_1\,x_3\,x_5\right)\right),\\
F&=c\,{x_0}^2\,{x_2}^3\,x_3+x_1\,{x_2}^3\,{x_3}^2\,x_5+x_0\,(c\,x_1\,{x_3}^3+3\,c\,x_1\,x_2\,{x_3}^2\,x_4\\
&\qquad+3\,c\,x_1\,{x_2}^2\,x_3\,{x_4}^2+{x_2}^3\,\left({x_3}^2+c\,x_1\,{x_4}^3+c\,x_1\,x_3\,x_5\right)),\\
G&=c\,{x_0}^2\,{x_2}^3\,x_3+x_1\,{x_2}^3\,{x_3}^2\,x_5+x_0\,(x_1\,{x_3}^3+(2+c)\,x_1\,x_2\,{x_3}^2\,x_4\\
&\qquad+(1+2\,c)\,x_1\,{x_2}^2\,x_3\,{x_4}^2+{x_2}^3\,\left({x_3}^2+c\,x_1\,{x_4}^3+c\,x_1\,x_3\,x_5\right)),\\
H&=c\,{x_0}^2\,{x_2}^3\,x_3+x_1\,{x_2}^3\,{x_3}^2\,x_5+x_0\,(x_1\,{x_3}^3+3\,x_1\,x_2\,{x_3}^2\,x_4\\
&\qquad+3\,x_1\,{x_2}^2\,x_3\,{x_4}^2+{x_2}^3\,\left({x_3}^2+x_1\,{x_4}^3+c\,x_1\,x_3\,x_5\right)).
\end{aligned}$$

7. Ultra-discretization

We denote the positive structure on χ as in the previous section by $\theta : T' := (\mathbb{C}^\times)^6 \longrightarrow \mathcal{V}_1$. Then by Corollary 2.7 we obtain the ultra-discretization $\mathcal{UD}(\chi, T', \theta)$, which is a Kashiwara's crystal. Now we show that the conjecture in [15] is correct and it turns out to be the following theorem.

Theorem 7.1. The crystal $\mathcal{UD}(\chi, T', \theta)$ as above is isomorphic to the crystal B_∞ of type $D_4^{(3)}$ as in Sect.4.

In order to show the theorem, we shall see the explicit crystal structure on $\mathcal{X} := \mathcal{UD}(\chi, T', \theta)$. Note that $\mathcal{UD}(\chi) = \mathbb{Z}^6$ as a set . Here as for variables in $\mathcal{X}$, we use the same notations $c, x_0, x_1, \cdots, x_5$ as for χ.

For $x = (x_0, x_1, \cdots, x_5) \in \mathcal{X}$, it follows from the results in the previous section that the functions wt_i and ε_i $(i = 0, 1, 2)$ are given as:

$$\mathrm{wt}_0(x) = 2x_0 - x_1 - x_3 - x_5,\ \mathrm{wt}_1(x) = 2(x_1 + x_3 + x_5) - x_0 - 3x_2 - 3x_4,$$
$$\mathrm{wt}_2(x) = 2(x_2 + x_4) - x_1 - x_3 - x_5.$$

Set

(7.1)
$$\begin{aligned}&\alpha := 2x_0 + 3x_2 + x_3, \quad \beta := x_1 + 3x_2 + 2x_3 + x_5, \quad \gamma := x_0 + x_1 + 3x_3,\\ &\delta := x_0 + x_1 + x_2 + 2x_3 + x_4, \quad \epsilon := x_0 + x_1 + 2x_2 + x_3 + 2x_4,\\ &\phi := x_0 + 3x_2 + 2x_3, \quad \psi := x_0 + x_1 + 3x_2 + 3x_4, \quad \xi := x_0 + x_1 + 3x_2 + x_3 + x_5.\end{aligned}$$

Indeed, from the explicit form of E as in the previous section we have

$$\mathcal{UD}(E) = \max(\alpha, \beta, \gamma, \delta, \epsilon, \phi, \psi, \xi),$$

and then

$$\begin{aligned}&\varepsilon_0(x) = \max(\alpha, \beta, \gamma, \delta, \epsilon, \phi, \psi, \xi) - (3x_0 + 3x_2 + x_3),\\ (7.2)\ &\varepsilon_1(x) = \max(x_0 - x_1, x_0 + 3x_2 - 2x_1 - x_3, x_0 + 3x_2 + 3x_4 - 2x_1 - 2x_3 - x_5),\\ &\varepsilon_2(x) = \max(x_1 - x_2, x_1 + x_3 - 2x_2 - x_4).\end{aligned}$$

Next, we describe the actions of $\tilde{e}_i$ $(i = 0, 1, 2)$. Set $\Xi_j := \mathcal{UD}(\mathcal{C}_j)|_{c=1}$ $(j = 1, \cdots, 5)$. Then we have

$$\begin{aligned}\Xi_1 &= \max(1 + x_0 - x_1, x_0 + 3x_2 - 2x_1 - x_3, x_0 + 3x_2 + 3x_4 - 2x_1 - 2x_3 - x_5)\\ &\quad -\max(x_0 - x_1, x_0 + 3x_2 - 2x_1 - x_3, x_0 + 3x_2 + 3x_4 - 2x_1 - 2x_3 - x_5),\\ \Xi_3 &= \max(1 + x_0 - x_1, 1 + x_0 + 3x_2 - 2x_1 - x_3, x_0 + 3x_2 + 3x_4 - 2x_1 - 2x_3 - x_5)\\ &\quad -\max(1 + x_0 - x_1, x_0 + 3x_2 - 2x_1 - x_3, x_0 + 3x_2 + 3x_4 - 2x_1 - 2x_3 - x_5),\\ \Xi_5 &= \max(1 + x_0 - x_1, 1 + x_0 + 3x_2 - 2x_1 - x_3, 1 + x_0 + 3x_2 + 3x_4 - 2x_1 - 2x_3 - x_5)\\ &\quad -\max(1 + x_0 - x_1, 1 + x_0 + 3x_2 - 2x_1 - x_3, x_0 + 3x_2 + 3x_4 - 2x_1 - 2x_3 - x_5),\\ \Xi_2 &= \max(1 + x_1 - x_2, x_1 + x_3 - 2x_2 - x_4) - \max(x_1 - x_2, x_1 + x_3 - 2x_2 - x_4),\\ \Xi_4 &= \max(1 + x_1 - x_2, 1 + x_1 + x_3 - 2x_2 - x_4) - \max(1 + x_1 - x_2, x_1 + x_3 - 2x_2 - x_4).\end{aligned}$$

Therefore, for $x \in \mathcal{X}$ we have

$$\tilde{e}_1(x) = (x_0, x_1 + \Xi_1, x_2, x_3 + \Xi_3, x_4, x_5 + \Xi_5),$$
$$\tilde{e}_2(x) = (x_0, x_1, x_2 + \Xi_2, x_3, x_4 + \Xi_4, x_5).$$

We obtain the action $\tilde{f}_i$ $(i = 1, 2)$ by setting $c = -1$ in $\mathcal{UD}(\mathcal{C}_i)$.

Finally, we describe the action of $\tilde{e}_0$. Set

$$\begin{aligned}
\Psi_0 &:= \max(2+\alpha,\beta,1+\gamma,1+\delta,1+\epsilon,1+\phi,1+\psi,1+\xi)\\
&\quad -\max(\alpha,\beta,\gamma,\delta,\epsilon,\phi,\psi,\xi)-1,\\
\Psi_1 &:= \max(1+\alpha,\beta,1+\gamma,1+\delta,1+\epsilon,\phi,1+\psi,1+\xi)\\
&\quad -\max(\alpha,\beta,\gamma,\delta,\epsilon,\phi,\psi,\xi)-1,\\
\Psi_2 &:= \max(1+\alpha,\beta,\gamma,1+\delta,1+\epsilon,\phi,1+\psi,1+\xi)\\
&\quad -\max(\alpha,\beta,\gamma,\delta,\epsilon,\phi,\psi,\xi)-1,\\
\Psi_3 &:= \max(2+\alpha,\beta,1+\gamma,1+\delta,1+\epsilon,1+\phi,1+\psi,1+\xi)\\
&\quad +\max(1+\alpha,\beta,\gamma,\delta,\epsilon,\phi,\psi,1+\xi)-\max(1+\alpha,\beta,\gamma,\delta,\epsilon,\phi,\psi,1+\xi)\\
&\quad -\max(1+\alpha,\beta,\gamma,1+\delta,1+\epsilon,\phi,1+\psi,1+\xi)-2,\\
\Psi_4 &:= \max(2+\alpha,\beta,1+\gamma,1+\delta,1+\epsilon,1+\phi,1+\psi,1+\xi)\\
&\quad -\max(1+\alpha,\beta,\gamma,1+\delta,1+\epsilon,\phi,1+\psi,1+\xi)-1,\\
\Psi_5 &:= \max(2+\alpha,\beta,1+\gamma,1+\delta,1+\epsilon,1+\phi,1+\psi,1+\xi)\\
&\quad -\max(1+\alpha,\beta,\gamma,\delta,\epsilon,\phi,\psi,1+\xi)-1,
\end{aligned}$$

where $\alpha,\beta,\cdots,\xi$ are as in (7.1). Therefore, by the explicit form of e_0^c as in the previous section, we have

$$\tilde{e}_0(x)=(x_0+\Psi_0,x_1+\Psi_1,x_2+\Psi_2,x_3+\Psi_3,x_4+\Psi_4,x_5+\Psi_5). \tag{7.3}$$

Now, let us show the theorem.
(*Proof of Theorem 7.1.*) Define the map

$$\begin{array}{cccc}\Omega\colon & \mathcal{X} & \longrightarrow & B_\infty,\\ & (x_0,\cdots,x_5) & \mapsto & (b_1,b_2,b_3,\overline{b}_3,\overline{b}_2,\overline{b}_1),\end{array}$$

by

$$b_1=x_5,\ b_2=x_4-x_5,\ b_3=x_3-2x_4,\ \overline{b}_3=2x_2-x_3,\ \overline{b}_2=x_1-x_2,\ \overline{b}_1=x_0-x_1,$$

and Ω^{-1} is given by

$$\begin{aligned}
&x_0=b_1+b_2+\frac{b_3+\overline{b}_3}{2}+\overline{b}_2+\overline{b}_1,\quad x_1=b_1+b_2+\frac{b_3+\overline{b}_3}{2}+\overline{b}_2,\\
&x_2=b_1+b_2+\frac{b_3+\overline{b}_3}{2},\ x_3=2b_1+2b_2+b_3,\ x_4=b_1+b_2,\ x_5=b_1,
\end{aligned}$$

which means that Ω is bijective. Here note that $\frac{b_3+\overline{b}_3}{2}\in\mathbb{Z}$ by the definition of B_∞. We shall show that Ω is commutative with actions of $\tilde{e}_i$ and preserves the functions wt_i and ε_i, that is,

$$\tilde{e}_i(\Omega(x))=\Omega(\tilde{e}_ix),\quad \mathrm{wt}_i(\Omega(x))=\mathrm{wt}_i(x),\quad \varepsilon_i(\Omega(x))=\varepsilon_i(x)\quad (i=0,1,2).$$

First, let us check wt_i: Set $b=\Omega(x)$. By the explicit forms of wt_i on $\mathcal{X}$ and B_∞, we have

$$\begin{aligned}
&\mathrm{wt}_0(\Omega(x))=\varphi_0(\Omega(x))-\varepsilon_0(\Omega(x))=2z_1+z_2+z_3+3z_4\\
&\quad=2(\overline{b}_1-b_1)+(\overline{b}_2-\overline{b}_3)+(b_3-b_2)+\frac{3}{2}(\overline{b}_3-b_3)=2(\overline{b}_1-b_1)+\overline{b}_2-b_2+\frac{\overline{b}_3-b_3}{2}\\
&\quad=2x_0-x_1-x_3-x_5=\mathrm{wt}_0(x),\\
&\mathrm{wt}_1(\Omega(x))=\varphi_1(\Omega(x))-\varepsilon_1(\Omega(x))\\
&\quad=b_1+(b_3-b_2+(\overline{b}_2-\overline{b}_3)_+)_+-(\overline{b}_1+(\overline{b}_3-\overline{b}_2-(b_2-b_3)_+)_+)\\
&\quad=b_1-\overline{b}_1-b_2+\overline{b}_2+b_3-\overline{b}_3=2(x_1+x_3+x_5)-x_0-3x_2-3x_4=\mathrm{wt}_1(x),\\
&\mathrm{wt}_2(\Omega(x))=\varphi_2(\Omega(x))-\varepsilon_2(\Omega(x))=b_2+\frac{1}{2}(\overline{b}_3-b_3)_+-\overline{b}_2+\frac{1}{2}(b_3-\overline{b}_3)_+\\
&\quad=b_2-\overline{b}_2+\frac{1}{2}(\overline{b}_3-b_3)=2(x_2+x_4)-x_1-x_3-x_5=\mathrm{wt}_2(x).
\end{aligned}$$

Next, we shall check ε_i:

$$\begin{aligned}
&\varepsilon_1(\Omega(x))=\bar{b}_1+(\bar{b}_3-\bar{b}_2+(b_2-b_3)_+)_+\\
&\ =\max(\bar{b}_1,\bar{b}_1+\bar{b}_3-\bar{b}_2,\bar{b}_1+\bar{b}_3-\bar{b}_2+b_2-b_3)\\
&\ =\max(x_0-x_1,x_0+3x_2-2x_1-x_3,x_0+3x_2+3x_4-2x_1-2x_3-x_5)=\varepsilon_1(x),\\
&\varepsilon_2(\Omega(x))=\bar{b}_2+\frac{1}{2}(b_3-\bar{b}_3)_+=\max(\bar{b}_2,\bar{b}_2+\frac{1}{2}(b_3-\bar{b}_3)_+)\\
&\ \max(x_1-x_2,x_1+x_3-2x_2-x_4)=\varepsilon_2(x).
\end{aligned}$$

Before checking $\varepsilon_0(\Omega(x))=\varepsilon_0(x)$, we see the following formula, which has been given in [13, Sect6].

Lemma 7.2. For $m_1,\cdots,m_k\in\mathbb{R}$ and $t_1,\cdots,t_k\in\mathbb{R}_{\geq 0}$ such that $t_1+\cdots t_k=1$, we have

$$\max\left(m_1,\cdots,m_k,\sum_{i=1}^{k}t_im_i\right)=\max(m_1,\cdots,m_k)$$

By the facts

$$\delta=\frac{2\gamma+\psi}{3},\quad \epsilon=\frac{\gamma+2\psi}{3}, \tag{7.4}$$

and Lemma 7.2, we have

$$\max(\alpha,\beta,\gamma,\delta,\epsilon,\phi,\psi,\xi)=\max(\alpha,\beta,\gamma,\phi,\psi,\xi). \tag{7.5}$$

Here let us see ε_0:

$$\begin{aligned}
&\varepsilon_0(\Omega(x))=-s(b)+\max A-(2z_1+z_2+z_3+3z_4)\\
&=-x_0+\max(0,z_1,z_1+z_2,z_1+z_2+3z_4,z_1+z_2+z_3+3z_4,2z_1+z_2+z_3+3z_4)-(\alpha-\beta)\\
&=-x_0+\max(-2x_0+x_1+x_3+x_5,-x_0+x_3,-x_0+x_1-3x_2+2x_3,\\
&\qquad\qquad -x_0+x_1-x_3+3x_4,-x_0+x_1+x_5,0)\\
&=-(3x_0+3x_2+x_3)+\max(x_1+3x_2+2x_3+x_5,x_0+3x_2+2x_3,x_0+x_1+3x_3,\\
&\qquad x_0+x_1+3x_2+3x_4,x_0+x_1+3x_2+x_3+x_5,2x_0+3x_2+x_3)\\
&=-(3x_0+3x_2+x_3)+\max(\beta,\phi,\gamma,\psi,\xi,\alpha).
\end{aligned}$$

On the other hand, we have

$$\varepsilon_0(x)=-(3x_0+3x_2+x_3)+\max(\alpha,\beta,\gamma,\delta,\epsilon,\phi,\psi,\xi).$$

Then by (7.5), we get $\varepsilon_0(\Omega(x))=\varepsilon_0(x)$.

Let us show $\tilde e_i(\Omega(x))=\Omega(\tilde e_i(x))$ ($x\in\mathcal{X}$, $i=0,1,2$). As for $\tilde e_1$, set

$$A=x_0-x_1,\ B=x_0+3x_2-2x_1-x_3,\ C=x_0+3x_2+3x_4-2x_1-2x_3-x_5.$$

Then we obtain $\Xi_1=\max(A+1,B,C)-\max(A,B,C)$, $\Xi_3=\max(A+1,B+1,C)-\max(A+1,B,C)$, $\Xi_5=\max(A+1,B+1,C+1)-\max(A+1,B+1,C)$. Therefore, we have

$$\begin{aligned}
&\Xi_1=1,\ \Xi_3=0,\ \Xi_5=0,\ \text{if } A\geq B,C\\
&\Xi_1=0,\ \Xi_3=1,\ \Xi_5=0,\ \text{if } A<B\geq C\\
&\Xi_1=0,\ \Xi_3=0,\ \Xi_5=1,\ \text{if } A,B<C,
\end{aligned}$$

which implies

$$\tilde e_1(x)=\begin{cases}(x_0,x_1+1,x_2,\cdots,x_5) & \text{if } A\geq B,C\\ (x_0,\cdots,x_3+1,x_4,x_5) & \text{if } A<B\geq C\\ (x_0,\cdots,x_4,x_5+1) & \text{if } A,B<C\end{cases}$$

Since $A=\overline{b}_1$, $B=\overline{b}_1+\overline{b}_3-\overline{b}_2$ and $C=\overline{b}_1+\overline{b}_3-\overline{b}_2+b_2-b_3$, we get ($b=\Omega(x)$)

$$\Omega(\tilde e_1(x))=\begin{cases}(\ldots,\bar b_2+1,\bar b_1-1) & \text{if } \bar b_2-\bar b_3\geq(b_2-b_3)_+,\\ (\ldots,b_3+1,\bar b_3-1,\ldots) & \text{if } \bar b_2-\bar b_3<0\leq b_3-b_2,\\ (b_1+1,b_2-1,\ldots) & \text{if } (\bar b_2-\bar b_3)_+<b_2-b_3,\end{cases}$$

which is the same as the action of $\tilde e_1$ on $b=\Omega(x)$ as in Sect.4. Hence, we have $\Omega(\tilde e_1(x))=\tilde e_1(\Omega(x))$.

Let us see $\Omega(\tilde e_2(x))=\tilde e_2(\Omega(x))$. Set

$$L=x_1-x_2,\quad M:=x_1+x_3-2x_2-x_4.$$

Then $\Xi_2=\max(1+L,M)-\max(L,M)$ and $\Xi_4=\max(1+L,1+M)-\max(1+L,M)$. Thus, one has

$$\begin{aligned}
&\Xi_2=1,\quad \Xi_4=0\quad \text{if } L\geq M,\\
&\Xi_2=0,\quad \Xi_4=1\quad \text{if } L<M,
\end{aligned}$$

which means

$$\tilde e_2(x)=\begin{cases}(x_0,x_1,x_2+1,x_3,x_4,x_5) & \text{if } L\geq M,\\ (x_0,x_1,x_2,x_3,x_4+1,x_5) & \text{if } L<M.\end{cases}$$

Since $L-M=x_2-x_3+x_4=\frac{\overline{b}_3-b_3}{2}$, one gets

$$\Omega(\tilde e_2(x))=\begin{cases}(\ldots,\bar b_3+2,\bar b_2-1,\ldots) & \text{if } \bar b_3\geq b_3,\\ (\ldots,b_2+1,b_3-2,\ldots) & \text{if } \bar b_3<b_3,\end{cases}$$

where $b=\Omega(x)$. This action coincides with the one of $\tilde e_2$ on $b\in B_\infty$ as in Sect4. Therefore, we get $\Omega(\tilde e_2(x))=\tilde e_2(\Omega(x))$.

Finally, we shall check $\tilde e_0(\Omega(x))=\Omega(\tilde e_0(x))$. For the purpose, we shall estimate the values $\Psi_0,\cdots,\Psi_5$ explicitly.

First, the following cases are investigated:

$$\begin{array}{ll}
\text{(e1)} & \beta > \alpha, \gamma, \delta, \epsilon, \phi, \psi, \xi, \\
\text{(e2)} & \beta \leq \phi > \alpha, \gamma, \delta, \epsilon, \psi, \xi \\
\text{(e3)} & \beta, \phi \leq \gamma > \alpha, \delta, \epsilon, \psi, \xi \\
\text{(e4)} & \beta, \gamma, \delta, \epsilon, \phi \leq \psi > \alpha, \xi \\
\text{(e4}') & \beta, \gamma, \epsilon, \phi, \psi \leq \delta > \alpha, \xi \\
\text{(e4}'') & \beta, \gamma, \delta, \phi, \psi \leq \epsilon > \alpha, \xi \\
\text{(e5)} & \beta, \gamma, \delta, \epsilon, \phi, \psi \leq \xi > \alpha, \\
\text{(e6)} & \alpha \geq \beta, \gamma, \delta, \epsilon, \phi, \psi, \xi.
\end{array}$$

It is easy to see that each of these conditions are equivalent to the conditions (E_1)-(E_6) in Sect.4, more precisely, we have (ei) $\Leftrightarrow$ (E_i) $(i = 1, 2, \cdots, 6)$, and that (e1)–(e6) cover all cases and they have no intersection. Note that the cases (e4') and (e4") are included in the case (e4) thanks to (7.4).

Let us show (e1)$\Leftrightarrow$ (E_1): the condition (e1) means $\beta-\alpha = -(2z_1+z_2+z_3+3z_4) > 0$, $\beta-\gamma = -(z_1+z_2) > 0$, $\beta-\delta = -(z_1+z_2+z_4) > 0$, $\beta-\epsilon = -(z_1+z_2+2z_4) > 0$, $\beta-\phi = -z_1 > 0$, $\beta-\psi = -(z_1+z_2+3z_4) > 0$ and $\beta-\xi = -(z_1+z_2+z_3+3z_4) > 0$, which is equivalent to the condition $z_1 + z_2 < 0$, $z_1 < 0$, $z_1 + z_2 + 3z_4 < 0$ and $z_1 + z_2 + z_3 + 3z_4 < 0$. This is just the condition (E_1). Other cases are shown similarly.

Under the condition (e1) ($\Leftrightarrow$ (E_1)), we have

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = \Psi_5 = -1, \quad \Psi_3 = -2,$$

which means $\tilde{e}_0(x) = (x_0 - 1, x_1 - 1, x_2 - 1, x_3 - 2, x_4 - 1, x_5 - 1)$. Thus, we have

$$\Omega(\tilde{e}_0(x)) = (b_1 - 1, b_2, \cdots, \overline{b}_1),$$

which coincides with the action of $\tilde{e}_0$ under (E_1) in Sect.4. Similarly, we have

$$\begin{array}{rrl}
\text{(e2)} & \Rightarrow & (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (0, -1, -1, -1, 0, 0) \\
& \Rightarrow & \tilde{e}_0(x) = (x_0, x_1 - 1, x_2 - 1, x_3 - 1, x_4, x_5), \\
& \Rightarrow & \Omega(\tilde{e}_0(x)) = (b_1, b_2, b_3 - 1, \overline{b}_3 - 1, \overline{b}_2, \overline{b}_1 + 1),
\end{array}$$

which coincides with the action of $\tilde{e}_0$ under (E_2) in Sect.4.

$$\begin{array}{rrl}
\text{(e3)} & \Rightarrow & (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (0, 0, -1, -2, 0, 0) \\
& \Rightarrow & \tilde{e}_0(x) = (x_0, x_1, x_2 - 1, x_3 - 2, x_4, x_5), \\
& \Rightarrow & \Omega(\tilde{e}_0(x)) = (b_1, b_2, b_3 - 2, \overline{b}_3, \overline{b}_2 + 1, \overline{b}_1),
\end{array}$$

which coincides with the action of $\tilde{e}_0$ under (E_3) in Sect.4.

$$\begin{array}{rrl}
\text{(e4)} & \Rightarrow & (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (0, 0, 0, -2, -1, 0) \\
& \Rightarrow & \tilde{e}_0(x) = (x_0, x_1, x_2, x_3 - 2, x_4 - 1, x_5), \\
& \Rightarrow & \Omega(\tilde{e}_0(x)) = (b_1, b_2 - 1, b_3, \overline{b}_3 + 2, \overline{b}_2, \overline{b}_1),
\end{array}$$

which coincides with the action of $\tilde{e}_0$ under (E_4) in Sect.4.

$$\begin{array}{rrl}
\text{(e5)} & \Rightarrow & (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (0, 0, 0, -1, -1, -1) \\
& \Rightarrow & \tilde{e}_0(x) = (x_0, x_1, x_2, x_3 - 1, x_4 - 1, x_5 - 1), \\
& \Rightarrow & \Omega(\tilde{e}_0(x)) = (b_1 - 1, b_2, b_3 + 1, \overline{b}_3 + 1, \overline{b}_2, \overline{b}_1),
\end{array}$$

which coincides with the action of $\tilde{e}_0$ under (E_5) in Sect.4.

$$\begin{aligned}(\mathrm{e}6)\quad &\Rightarrow\quad (\Psi_0,\Psi_1,\Psi_2,\Psi_3,\Psi_4,\Psi_5)=(1,0,0,0,0,0)\\ &\Rightarrow\quad \tilde{e}_0(x)=(x_0+1,x_1,x_2,x_3,x_4,x_5),\\ &\Rightarrow\quad \Omega(\tilde{e}_0(x))=(b_1,b_2,b_3,\overline{b}_3,\overline{b}_2,\overline{b}_1+1),\end{aligned}$$

which coincides with the action of $\tilde{e}_0$ under (E_6) in Sect.4. Now, we have $\Omega(\tilde{e}_0(x)) = \tilde{e}_0(\Omega(x))$. Therefore, the proof of Theorem 7.1 has been completed. □

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